NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XII BATON ROUGE, LA., NOVEMBER, 1937 No. 2

PUBLISHED BY LOUISIANA STATE UNIVERSITY

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

1. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the Magazine.

Papers intended for the Teacher's Department, Department of History of Mathematics, Reviews and Abstracts or Problem Department should be sent to the respective Chairmen.

Committee on Algebra and Number Theory: L. E. Bush, W. Vann Parker.

Committee on Analysis and Geometry: W. E. Byrne, Wilson L. Miser, Dorothy McCoy, H. L. Smith.

Committee on Teaching of Mathematics: Joseph Seidlin, James McGiffert.

Committee on Statistics: C. D. Smith, Irby C. Nichols.

Committee on Mathematical World News: L. J. Adams.

Committee on Reviews and Abstracts: P. K. Smith.

Committee on Problem Department: R. C. Yates.

Committee on Humanism and History of Mathematics: G. Waldo Dunnington.

Published by the Louisiana State University Press

Club Rates
May Be Had
on Application

Subscription, $1.50
Per Year in Advance
Single Copies, 20c.

Vol. XII BATON ROUGE, LA., NOVEMBER, 1937 No. 2

Published 8 Times Each Year by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

All Business Communications should be addressed to the Editor and Manager P. O. Box 1322, Baton Rouge, La.

This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Mathematics

The present rapidly widening use of mathematical technique and formula in fields ordinarily viewed as non-mathematical leads to some speculation regarding the probable extent to which research in nearly all fields of knowledge shall come to use some form of mathematics. If it is to be solid, the basis of this speculation must embrace consideration of the *pattern aspects* of a mathematical system. By "pattern aspects" is meant: those inherent structural features of a system that, when taken along with some of its immediate logical implications, make it an ideal model for fields in which precise knowledge is desired. An example of the latter is the field of economics. Within recent decades there has been taking place a steady re-formulation of economic theory in terms of mathematics. For many other examples one has but to read such an article as "*Vitalizing Mathematics*" published in the October issue of this journal by Will E. Edington of De Pauw University.

Given any domain of thought in which the fundamental objective is a knowledge that transcends mere induction or mere empiricism, it seems quite inevitable that its processes should be made to conform closely to the pattern of a system free of ambiguous terms, symbols, operations, deductions; a system whose implications and assumptions are unique and consistent; a system whose logic confounds not the necessary with the sufficient where these are distinct; a system whose materials are abstract elements interpretable as reality or unreality in any forms whatsoever provided only that these forms mirror a thought that is pure. To such a system is universally given the name MATHEMATICS.

S. T. SANDERS.

Imaginary Orders

By JAMES BYRNIE SHAW
University of Illinois

The human intellect prefers order to chaos. Indeed it was pointed out many years ago by C. S. Peirce that even if we live in a world of pure chance and utter chaos, we would nevertheless report uniformities and laws for it. We would select our facts according to their significance and so that they would exhibit order. Some facts would have a higher value for us than others and from these we would deduce relations which we perhaps had ourselves unconsciously put in. Order is one of the elements of beauty, along with rhythm, design and harmony. In science order is usually called law. When a scientist studies the spectrum of an element and finds what he calls energy levels, and then describes the phenomena in terms of wave analysis, or quantum theory, he is studying order. So in numbers, ideal creations of our own, we find order, the way numbers are built and put together. The first numbers were the natural numbers, 1, 2, and so on. These were then put together in ratios. And eventually there came into being entirely new numbers which, at first, did not seem to be connected with objects.

Among these numbers were some which puzzled the early mathematicians very much. We may date the beginning of algebra as different from arithmetic from the invention of the negative number, due to some unknown genius. Later it became necessary to create a square root for every number, whether positive or negative. The square roots of negatives are still called *imaginary*. In the beginning these were called fictitious numbers, mere empty symbols. This notion is still current among non-mathematicians, although it became extinct towards the end of the sixteenth century, when Bombelli inquired how the sum of two non-existent numbers could give an existent number. This question arose in solving the equation

$$x^3 = 15x + 4.$$

A formula that had been recently announced by Cardan gave as one root

$$\sqrt[3]{(2+11\sqrt{-1})} + \sqrt[3]{(2-11\sqrt{-1})}$$

which reduces to $2+\sqrt{-1}+2-\sqrt{1}=4$. We have taken a cube root of two imaginary numbers giving two imaginary numbers whose sum is 4, a respectable real number. If we consider that 4 exists we can scarcely deny that the imaginary numbers exist. The other two roots of the cubic equation would be even more tangled up with square roots of negatives, but when finally reduced are $-2+\sqrt{3}$ and $-2-\sqrt{3}$. Since that day mathematicians in general have considered imaginary numbers to be just as real as any others. The term "imaginary" has become just a technical name. We have extended the field of numbers to include what we call "complex" numbers.

The complex numbers enable us to find solutions for all quadratic equations. For instance the roots of the equation $x^2-4x+52=0$ are the two complex numbers $2+4\sqrt{3}i$ and $2-4\sqrt{3}i$, where i is used to represent $\sqrt{-1}$. These two roots seem to be made up of several numbers, $2, 4, \sqrt{3}$ and i. We could condense these into 2 and $\sqrt{48}i$. These two cannot be combined so as to give a single number of the old kind. In fact there is a new character in evidence, for we cannot place a number of this kind between two integers so that it is greater than one and less than the other. We can do this with "real" numbers, even if they are Galois irrationals. We must not make the mistake of thinking such a number is really duplex however or that we must define it by 2 and $4\sqrt{3}$. If we represent the complex cube root of unity by ω the roots would be represented by

$$2\omega-6\omega^2 \text{ and } 2\omega^2-6\omega$$

defined now by 2 and -6. In fact each root is a single entity, and that we have to use two ordinary numbers to define it is similar to using 1 and 2 to define "half".

After some centuries these numbers were made useful by Steinmetz in the theory of alternating currents. In direct currents we have current, resistance, and electromotive force, but in alternating currents these become much more complex. We have in each case an entity which we use two numbers to express. If the resistance of such a circuit is r, the inductance is L, the frequency f, then $2\pi fL$ is the reactance, x. The impedance which takes the place of resistance is then $Z=r-xi$. Also in a power line there is at any particular point a power current u and a wattless current v, the entire current being $I=u+vi$. At the same point there will be an electromotive force of a power kind p and one of a wattless kind q, the complete electromotive force represented by $E=p+qi$. We may now state the relation between these in an equation that reduces to Ohm's law for direct currents $E=ZI$. The power of the current will be $E'I$ where E' is $p-qi$,

the conjugate of E. If we stop to remember that our description of these magnitudes by a pair of numbers in each case is artificial, since the actual phenomenon is unity, we will see that the complex numbers themselves are really not complex. Since now complex numbers may be used for practical work, indeed simplifying the problems of alternating currents enormously, it is all the more evident that they are not imaginary any more than everyday numbers. They were also used early in the preceding century to represent segments of lines in a plane, and various combinations of such segments in geometrical relations. A calculus of lines was developed. The first use of this kind was due to Wessel, a Danish Surveyor.

About the time Galois died, a very brilliant genius was looking for other imaginary numbers. He was Sir William Rowan Hamilton, Royal Astronomer of Ireland. He is famous for many important theorems. His biography takes up three large volumes. Many of his papers are still unpublished, seventy years after his death. If we look at what he was doing in the study of numbers we will see a whole universe of new numbers. He first asked himself the question: Are there other roots of unity like $\sqrt{-1}$ which could be used in geometry, particularly the geometry of space of three dimensions? He found an answer to this question and at the same time opened a new branch of mathematics in which all kinds of imaginary numbers exist and combine in new orders quite different from those known before his time. Just as Galois showed there is an infinity of rhythms so Hamilton showed there is an infinity of orders of combination.

To follow the line of thinking of Hamilton, let us notice first that if the product of two lines represented by complex numbers is to be a line, its length must be the product of the lengths of the two factors. This is shown in a formula as old as Diophantos, sometimes called the Father of Algebra. It is

$$(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2.$$

If we notice that if a, b are sides of a right triangle, the first parenthesis is the square of the length of the hypotenuse, and likewise if c, d are sides of another right triangle, the second parenthesis is the square of the length of its hypotenuse, and if we consider that the right hand side is the square of the length of a hypotenuse, the sides would have to be $ac-bd$, $ad+bc$. But if we multiply together the two complex numbers $a+bi$, $c+di$ the product is $(ac-bd)+(ad+bc)i$. Consequently we see that by using complex numbers we arrive at the two parts of the right hand side of the formula. The expressions a^2+b^2, c^2+d^2 are

called the ***norms**** of the numbers $a+bi$, $c+di$. We see then that for complex numbers the product of their norms is the norm of their product. This is remarkable because the list of algebraic expressions satisfying such a law is not very large. Another such formula was know to Euler:

$$\begin{aligned}(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2)\\ =(ap-bq-cr-ds)^2+(aq+bp+cs-dr)^2\\ +(ar-bs+cp+dq)^2+(as+br-cq+dp)^2.\end{aligned}$$

If we make d, s both zero we have the problem Hamilton was considering, and the sum of three squares in each case is the diagonal squared of a rectangular block in space. Hamilton found that to arrive at what he wanted he must use four numbers in each case. Hamilton said: Can we arrive at the expressions on the right by a multiplication of two quite complex numbers? If we introduce new imaginaries i, j, k, and write

$$\begin{aligned}A&=a+bi+cj+dk\\ P&=p+qi+rj+sk\\ Z&=(ap-bq-cr-ds)+(aq+bp+cs-dr)i\\ &\quad+(ar-bs+cp+dq)j+(as+br-cq+dp)k\end{aligned}$$

the problem is to adjust the product AP so that it will be Z. We assume that we can write out this product term by term just like ordinary algebraic polynomials. The resulting form must be valid for any values of the letters a, b, c, d, p, q, r, s. From the terms containing ap, bq, cr, ds we see that we must have $ii=-1$, $jj=-1$, $kk=-1$. The terms with a or p appear as they should. We have left only six terms to account for. If we notice the terms containing br we see that we must have $ij=k$. From cq, $ji=-k$. Similarly from the others we have $jk=i=-kj$, $ki=j=-ik$. With these laws of order we have satisfied everything. The results will be more easily seen in a table of combination or multiplication of the imaginary numbers i, j, k:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

*Hamilton and Cayley used the term "modulus".

The first factor is in the lefthand column, the second factor in the top row, the product in the table. These laws give us $AP=Z$. Again the products of the norms of the complex numbers is the norm of their product. Numbers of this type Hamilton called *Quaternions*. He wrote many articles and two large volumes about them and their uses in geometry. They have since been used extensively in physics, furnishing very direct and easily managed expressions. We notice that i, j, k are square roots of -1. The surprising law however is one that startled Hamilton, that is we must have $ji=-ij$, $jk=-kj$, $ik=-ki$. It makes a difference which factor comes first. We could find easily that AP is not equal to PA nor are they in general opposites. This is a new type of order. We say: these products are not usually commutative. These numbers will include ordinary complex numbers, for if the coefficients of two of the imaginaries are kept zero the quaternions resulting are simply common complex numbers. This shows us that quaternions are actually extensions of complex numbers, which are themselves extensions of real numbers. Another property which Hamilton insisted upon was that any quaternion could be divided by any other not entirely zero, with a unique quotient. This property follows from the law of norms, for the norm cannot vanish for "real" coefficients unless the quaternion vanishes.

Hamilton showed that if we draw two segments from the same vertex in space the quotient (as defined) will be a quaternion. A quaternion is also a single number but under our arithmetic system will need four ordinary numbers to define it, or more precisely, an equality between two quaternions will furnish four equalities between real numbers. If we notice that we can write $Q=a+bi+cj+dij$, we might think that a quaternion is similar to a Galois number like $a+b\sqrt{p}+c\sqrt{q}+d\sqrt{pq}$, but we can express the three square roots in this number as rational polynomials in a single number $\sqrt{p}+\sqrt{q}=r$, while if we set $s=xi+uj$ no power of s will contain ij. Quaternions are then extensions in a new direction of Galois irrationals. Numbers like i, j, k are called *vids* or sometimes qualitative units. They are not the only quaternions which are square roots of -1, for if we set $\alpha=li+mj+nk$, then we will have $\alpha^2=-1$ whenever we have $l^2+m^2+n^2=1$. There will thus be a double infinity of such square roots of unity. Also it is easy to find an infinity of systems like i, j, k. We let $\beta=l'i+m'j+n'k$ and $\gamma=l''i+m''j+n''k$. Then if we also have besides the equation above, the further equations $l'^2+m'^2+n'^2=1$, $ll'+mm'+nn'=0$, $l''=mn'-m'n$, $m''=nl'-n'l$, $n''=lm'-l'm$, then the three numbers α, β, γ will have a multiplication table similar to that of i, j, k.

To make use of quaternions in physics we must find phenomena whose laws follow those of quaternions, that is, have the same order. One notable instance is the restricted relativity theory. Details may be found in Silberstein's Theory of Relativity. The fact that a single quaternion may be expressed in different ways furnishes an expression for one of the fundamental laws of the physical theory. We find a physical system like i, j, k in the electric, magnetic, and Poynting vectors of a moving electromagnetic wave-front. We also have suggested to us that if any phenomenon needs several ordinary numbers to define it, it may be expressible by a single hypernumber. Indeed this is what has happened in the modern theory of the atom. We will return to this later. The motion of a rigid body needs a system of eight units to handle it.

The next step in this direction was made by Cayley. Early in the century Degen had shown that the sum of eight squares multiplied by the sum of eight squares, could be written as the sum of eight squares. This is again a form which follows the law of norms. After some study Cayley found that a system of hypernumbers existed which would give the proper forms. We will economise space by giving his table directly, for from it can be written out at once the proper forms for the eight squares into which the product could be decomposed. The first set of numbers we will indicate by S, T, U, V, W, X, Y, Z and the second set by s, t, u, v, w, x, y, z. The vids will be taken as $1, i, j, k, a, b, c, d$. The table is then given as follows:

		s	t	u	v	w	x	y	z
		1	i	j	k	a	b	c	d
S	1	1	i	j	k	a	b	c	d
T	i	i	-1	k	$-j$	b	$-a$	$-d$	c
U	j	j	$-k$	-1	i	c	d	$-a$	$-b$
V	k	k	j	$-i$	-1	d	$-c$	b	$-a$
W	a	a	$-b$	$-c$	$-d$	-1	i	j	k
X	b	b	a	$-d$	c	$-i$	-1	$-k$	j
Y	c	c	d	a	$-b$	$-j$	k	-1	$-i$
Z	d	d	$-c$	b	a	$-k$	$-j$	i	-1

If we follow throughout the table the terms that will contain b we will have one of the eight squares making up the product: $(Sx+Tw-Uz+Vy-Wt+Xs-Yv+Zu)$. The others can be easily written down. Hypernumbers like these Cayley called *octaves*. A new way of ordering products appears here. If we find $(ik)b$ it is $-jb=-d$, while $i(kb)=i(-c)=+d$. It makes a difference then in this algebra not

only whether we write AB or BA, but also whether we write $(AB)C$ or $A(BC)$. The "associative" property has disappeared. We find that the triples i,j,k; i,a,b; i,d,c; j,b,d; j,a,c; k,a,d; k,c,b are associative, and any other triple of vids is non-associative. We notice further that the part of the table arising from i, j, k is the table for quaternions, which is included in the octave table, just as part of the quaternion table, that containing 1 and i, is the table for complex numbers. That is, we have extended real numbers into complex numbers, the ***complex field***, then extended that into the quaternion field, and finally that into the Cayley octave field. These are ***fields*** because in them we can add, subtract, multiply, or divide. It is a very different extension from the Galois extension, for every number introduced by that is at the most complex. We have passed out of the linear world of numbers into worlds of two, four, and eight dimensions in numbers. We have discovered ways to represent much more complicated entities in physics and chemistry as well as in geometry. Indeed by continuing to expand we find we are able ultimately to represent the complicated atoms of modern chemistry. The dream of Pythagoras is coming true and the universe is expressible mathematically—so far as certain aspects of it are concerned.

Since we have seen commutativity and associativity disappear we may ask: are there sets of vids for which other properties disappear? For instance in the cases considered the vids were roots of unity. This was a property that interested Hamilton, and he wrote papers on other "roots of unity", notably cube roots and fifth roots, in tables which were really those of the tetrahedral, octahedral, and icosahedral groups. And Cayley considered any group table to define roots of unity. We cannot follow this line however. Instead let us go back to the fundamental identity:

$$(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$$

If we consider that the right hand side is made up from the two expressions of a product of two hypernumbers, and try to take them in the form $a+bi$, $c+di$, we would have to have $i^2=1$, and also $bic=-bci$. Since we wish to be free to place the coefficients on either side of a vid, this gives a contradiction. We would have to try a little different attack, and indeed if we use two vids we solve the problem, that is, if

$$x=ai+bj, \qquad u=ci+dj, \qquad z=(ac+bd)i+(ad-bc)j$$

we have the table for the two vids as follows:

$$\begin{array}{c|cc} & i & j \\ \hline i & i & j \\ j & -j & i \end{array}$$

No combination of hypernumbers from this table can be a root of unity. Neither are they commutative nor associative. They do satisfy equations however, as

$$i^2-i=0 \quad j\cdot j^2+j=0 \text{ and if } X=xi+yj,\ X\cdot X^2-2xX^2+(x^2+y^2)X=0.$$

The norm of X is x^2+y^2 and the law of norms holds. It is quite fundamental. If we study, as Cayley did, what algebras with two vids we can have with a law of norms, we may simplify the problem by remembering that a norm does not have to be a sum of squares. We may take as the norm of $xi+yj$, xy. There are then four types of algebras for this norm:

$$\begin{array}{c|cc} & i & j \\ \hline i & i & 0 \\ j & 0 & j \end{array} \qquad \begin{array}{c|cc} & i & j \\ \hline i & 0 & i \\ j & j & 0 \end{array} \qquad \begin{array}{c|cc} & i & j \\ \hline i & 0 & j \\ j & i & 0 \end{array} \qquad \begin{array}{c|cc} & i & j \\ \hline i & j & 0 \\ j & 0 & i \end{array}$$

Again new features appear, for in the second and third we see that i, j are now roots of zero. This is startling, for zero is becoming as important as any other number. Nature has come to love a vacuum. In the first and fourth the products ij, ji are zero when neither is zero. They are said to be ***nilfactorial***, and a root of zero is called a ***nilpotent***. In the first the squares of i,j are i,j and they are called ***idempotent***. In the fourth we have $(ii)=j$, $i(ii)=0=(ii)i$, $(ii)(ii)=i$. These are certainly new orders for multiplication. Also in the first we notice $(i+j)i=i=i(i+j)$, and $(i+j)j=j=(i+j)$, so that $i+j$ behaves exactly like the ordinary unit 1, and we find that this "unit" may be broken up into two idempotent parts. Further if we let $x=ai+bj$ and remember we may write now

$$a=a(i+j), \quad b=b(i+j),$$

then $$x-a=(b-a)j, \quad x-b=(a-b)i,$$

and $$(x-a)(x-b)=0$$

This enables us to see that a long-standing paradox is finally solved. We are accustomed to the statement that the solution of this quadratic is ambiguous, either of two roots being a solution. But we see that we should say "the solution is $ai+bj$" and is unique. The equation defines a single number. Likewise an equation of degree N defines a single hypercomplex number.

In the tables above we find in the first three idempotents: $i,j,i+j$; for the second i,j are nilpotents and $i+j$ is idempotent; the same is true for the third; for the fourth, if we let ω be the primitive cube

root of 1, we have the three idempotents: $i+j$, $\omega i+\omega^2 j$, $\omega^2 i+\omega j$, the last being the product of the first two, as well as being the negative of their sum.

We will find more general forms for algebras of this kind if we choose a different quadratic for the norm. Let us select for the norm of $xi+yj$ the form $(b^2+d^2)(x^2+y^2)$. For two such hypernumbers, the other being $ui+vj$, if we write the product as

$$(bxu-dxv-dyu-byv)i+(dxu+bxv+byu-dyv)j$$

we find the table

$$\begin{array}{c|cc} & i & j \\ \hline i & bi+dj & -di+bj \\ j & -di+bj & -bi-dj \end{array}$$

We may find more general tables similarly for the other three. In these four cases we can arrive at the two terms of the product in another way by using complex numbers alone. Let $A=x+y\sqrt{-1}$, $B=u+v\sqrt{-1}$, $C=b+d\sqrt{-1}$. Indicate the conjugates of these with an accent, and define as "product" the four forms

$$A{*}B=CAB, \quad A{*}B=CA'B', \quad A{*}B=CAB', \quad A{*}B=CA'B.$$

The real and the imaginary coefficients of these products will be the coefficients of i,j for the four algebras above. The first two we can show will be associative. These examples show that for the same formula for a law of norms we may devise different algebras, and these algebras may not be expressible in terms of one another. We could have found more general norms for four letters than those of quaternions in this way. Let us select a fixed quaternion A, and indicate the conjugate of a quaternion by an accent. Then if we select as the norm of Q, $AA'QQ'$, we find four new algebras by using as coefficients of their vids i,j,k,l the four coefficients of the quaternion product determined by

$$Q{*}R=QAR, \quad Q{*}R=QAR', \quad Q{*}R=Q'AR, \quad Q{*}R=Q'AR'.$$

We can do the same with Cayley's octaves. Indeed to reach the formula of Degen we must use even more general forms than these. We can discover new norms in this manner.

We return now to other forms of norms long known. One of the simplest is the determinant. The product of two determinants of second order is a determinant of second order. That is

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} \begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{vmatrix}$$

If we use these as norms for hypernumbers, say

$$A = pi + qj + rk + sl,$$

$$B = wi + xj + yk + zl,$$

$$AB = Z = (pw + qy)i + (px + qz)j + (rw + sy)k + (rx + sz)l$$

we would have the table

	i	j	k	l
i	i	j	0	0
j	0	0	i	j
k	k	l	0	0
l	0	0	k	l

It will be easier to see the order in this table if we change the notation, using e_{ab} as the vid that goes with the coefficient situated in the position, a,b. In the determinant form, going with q,x would be the vid e_{12}. Then we have the table summed up in the equations

$$e_{ab}e_{bc} = e_{ac}$$

$$e_{ab}e_{cd} = 0 \text{ if } b \neq c.$$

An algebra like this is called a *quadrate algebra*. We may connect a quadrate algebra with a determinant of any order, the two equations given above serving to define the multiplication table for any order. They were studied by Sylvester and by C. S. Peirce. The latter showed that any linear associative algebra would be a sub-algebra of a proper quadrate algebra. The study of matrices reduces at once to the study of quadrate algebras. We can also find more complicated forms in these cases by using a fixed determinant corresponding to a fixed hypernumber, and using the symmetric determinant form as a conjugate.

In 1925 Heisenberg developed a new system of mechanics called matric mechanics. His purpose was to find equations for determining the energy levels of atoms and the intensities and frequencies of their spectral lines. The coefficients standing in the array which gives either the determinant or the hypernumber were the forms

$$A_{nm}e^{2\pi i\nu_{nm}t}$$

in the n,m position, where ν_{nm} is the frequency for a transition from the nth energy level to the mth, A_{nm} is the corresponding amplitude or displacement of a virtual oscillator. The numbers down the diagonal

correspond to stationary states of the atom, numbers below the diagonal correspond to emission states, those above to absorption states. The whole theory is of course rather complicated, as are the phenomena. But it is very remarkable that algebras like these should have become necessary in physics, when their origin was simply in the fascinating play of the minds of the creative mathematicians of many years ago.

We must now ask if the law of norms also disappears, and the answer is: yes. Algebras giving a law of norms are relatively few compared to all algebras. For only two vids Cayley showed that there are eleven classes of algebras, of which only two give a law of norms, each having two non-equivalent forms. We will give directly the Cayley forms. In each case the vids are i,j and only the table need be written. The classes are as follows:

I.

Three idempotents, no nilpotents in the algebra.

$$\begin{array}{|cc} \hline i & ci+dj \\ ei+fj & j \end{array}$$

$c+e\neq 1\neq d+f$, $(c+e)(d+f)\neq 1$ The third idempotent is

$$\frac{(1-c-e)i+(1-d-f)j}{1-(c+e)(d+f)}$$

II.

Two idempotents, one nilpotent. As above but with $(c+e)(d+f)=1$ The nilpotent is

$$-(c+e)i+j$$

III.

One idempotent, two nilpotents.

$$\begin{array}{|cc} \hline 0 & ci+dj \\ ei+fj & 0 \end{array}$$

The idempotent is

$$\frac{(c+e)i+(d+f)j}{(c+e)(d+f)}$$

IV.

In this case we have one double idempotent and one single idempotent. By double idempotent we mean a hypernumber x such that $x^2=tx$ and also for a determinate y we will have $xy+yx=ty+zx$. z and t are ordinary real or complex numbers. A double idempotent arises from the equation in the most general form for a dual algebra which determines the conditions for idempotents. Such equation is a cubic and if it has a double root there will be a double idempotent. The table is

$$\begin{array}{|cc} \hline i & ci+dj \\ ei+(1-d)j & j \end{array}$$

The single idempotent is j, the double is i, and the accompanying hypernumber is any hypernumber of the algebra except i, for if

$$y=ui+vj,\ iy+yi=y+(u+vc+ve)i.$$

V.

A single idempotent j, and a double nilpotent i, with accompanying hypernumber $y=i+vj$,

$$\begin{array}{|cc} \hline 0 & ci+dj \\ ei-dj & j \end{array} \qquad\qquad c+e\neq 1.$$

VI.

A single nilpotent and a double idempotent.

$$\begin{array}{|cc} \hline i & ci+dj \\ ei+(1-d)j & 0 \end{array}$$

VII.

A single nilpotent and a double nilpotent.

$$\begin{array}{|cc} \hline 0 & ci+dj \\ ei-dj & 0 \end{array}$$

VIII.

A triple idempotent. The extra condition is that there is a hypernumber z such that

$$zi+iz+2y^2=tz+t'y+t''i. \text{ with } i^2=ti,\ iy+yi=ty+t'i.$$

$$\begin{array}{|cc} i & ci+dj \\ ei+(1-d)j & gi+(c+e)j \end{array}$$

$$y=i+uj, \quad z=vi+wj.$$

IX

A triple nilpotent.

$$\begin{array}{|cc} 0 & ci+dj \\ ei-dj & gi+(c+e)j \end{array}$$

X.

If there are more than three idempotents there is an infinite number.

$$\begin{array}{|cc} i & ci+dj \\ -ci+(1-d)j & 0 \end{array}$$

Every hypernumber $i+uj$ is idempotent.

XI.

An infinite number of nilpotents.

$$\begin{array}{|cc} 0 & ci+dj \\ -ci-dj & 0 \end{array}$$

Every hypernumber of the algebra is nilpotent.

We now have seen a variety of algebras, with many roots of unity, roots of zero, neither roots of unity nor of zero, idempotents, nilpotents, single and multiple. The world of number is vastly more complicated than even the great Pythagoras could dream of. We are in a position to see that we can start with any number of vids and write the table of their products, each product being any linear form in the vids. The coefficients may be chosen from any field, or domain of integrity. Hamilton saw the general problem but it was left to successors to carry on the study. From an abstract point of view it becomes immaterial whether the vids represent quality signs or represent operators or even expressions which are complicated. Abstract algebra has been studied a good deal, although much of this study has consisted in labeling anew what already had been developed in some form. One notable result has been the inclusion of formal logic as merely a special form of algebra, and thence the creation of new logics, and new modes of reasoning.

An algebra is called a ***division algebra*** if any hypernumber in it can be divided by any other giving a unique quotient, except that 0 is not a divisor. For instance the algebra

$$\begin{array}{|cc} i & j \\ -j & i \end{array}$$

is a division algebra. The quotient of $xi+yj$ by $ui+vj$ on the right is

$$\frac{(xu-vy)i+(xv+uy)j}{u^2+v^2}$$

which is definite as long as u,v are real. A division algebra is a field. In case we cannot divide by every number an algebra may be called a "ring", a term of undesirable connotation and quite superfluous.

The applications of algebras remain largely to be worked out. They are very useful in geometry. It was Descartes who first pointed out the equivalence of geometry and algebra, and this equivalence is far wider than he imagined. Most modern geometry is really algebra under a different terminology.

We may classify rhythms by classifying the numbers connected with them. Every prime starts a new rhythm and every primitive Galois irrational starts more complicated rhythms. So we may classify orders by classifying the algebras connected with them. If we combine hypernumbers with rhythms of the most general kinds we have structures of great complexity but great beauty.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

The Fourth International History of Science Congress, Prague, Sept. 22-27, 1937

By DR. JÓZSEF JELITAI (v. Woyciechowsky)
Budapest, Hungary

In the plenary session of September 22 Gino Loria delivered the chief lecture: ***The role of geometric representation of magnitudes in the different epochs of the history of mathematics.*** The content is thus outlined: (Introduction. The representation of numbers by segments in Euclid. Researches of Huyghens and of Fermat on the rectification of curves and on the projection of surfaces; enunciations of the results under a geometric form. Coordinates and their use in analysis. Extraordinary curves and the limitation of their use in the analysis of geometric representation. Conclusion: some general remarks relative to researches in the history of science; its goal and different character.)

The mathematics section held three sessions: September 23, 24, and 27, 10:15 to 12:30. The chairmen were Birkenmajer, Bortolotti, and Vollgraff. The meetings were held at the Université Charles IV, Prague II, Albertov 6, on the second floor right. The weather was cool and rainy, the buildings were heated. The section meetings were well attended, with 40 to 60 present. A total of 24 papers were given; 40 papers had been announced, but twelve of these were missing. On September 27 at 11:30 a. m. in the final plenary session six Czecho-Slovakian members of the Congress were still waiting to give their papers, and there was insufficient time to hear them. One would have had to hold at least four or five sessions, instead of only three, in order to take up all papers.

SEPTEMBER 23

Chairman: Birkenmajer

1. Mrs. Dorothea Waley Singer, Kilmarth, Parish Cornwall: ***The Cosmology of Giordano Bruno.*** (Bruno was neither astronomer nor

metaphysician. His cosmology led him to a new philosophy and ethics. Important elements in Bruno's cosmology: Infinity and Eternity of the Universe; Innumerable Worlds; Cosmic Metabolism; Immanent Necessity; The Subject-Object Relation, culminating in Identity; The Monad; the Universe comprises a sort of "discrete continuity". Universal relativity within the World Soul.

2. *The History of Mathematics in Hungary Before 1830.* (Cf. Dec. issue of the NATIONAL MATHEMATICS MAGAZINE), by József Jelitai. Discussion after the paper: Chairman Birkenmajer—"The mathematical manuals by Paulus Makó de Kerekgede and the various works and tables of the University Press Tyrnaviae were also circulated in Poland and thus the research of Jelitai also has much of interest for Poland". Loria—"It is remarkable that Hungary, although a small country, can boast of a world famous mathematician's name such as Bolyai." He welcomes the paper and hopes that Jelitai will write the history of mathematics in Hungary.

3. Prof. Alex. Birkenmajer (Cracow): *Were the Babylonians acquainted with logarithms?* A Berlin mathematical tablet (Vat. 8528) has suggested to Neugebauer the question as to how the Babylonians resolved the exponential equation $2^x=32$. The learned historian is inclined to believe that they possessed the notion of a logarithm and that they also had tables analogous to our tables of logarithms. The author of the present paper cannot share these opinions; he believes rather that the problem in question was resolved by a procedure entirely elementary. Discussion: Bortolotti also considers Neugebauer's interpretations in erorr.

4. Otto Blüh, agrégé, Prague: *Physics and World-view as Actual and Historical Problem.* There are two schools which are pitted one against the other: the school of "realists" desires to generalize the essence of laws recently found in physics, while the school of "neopositivists" ("Wiener School") desires to apply the "physical method" recently formed also in part from the domain of special science.

5. Otto Seydl, astronomer, Prague: *The beginnings of the Prague Observatory.* Founded 1721. Father Joseph Stepling, S. J. (1716-1778) bought with his own means some instruments for 4000 florins; he founded a physics laboratory at the university and published 27 papers and longer disquisitions on mathematics, physics and astronomy.

6. Prof. Jan Vojtech, Prague: *Bolzano's achievements in geometry.* Bernard Bolzano (1781-1848) having been proposed for the chair of mathematics as successor of S. Vydra at the University of Prague, he was in 1805 named professor of the science of religion in this school,

but toward the end of 1819 he was deprived of this chair on account of his independent character. He gave an outline of the theory of the straight line, a definition of congruence from the point of view of logic. Very interesting, but unfortunately incorrect is his deduction of formulas for the rectification of a line and the projection of a plane according to similarity.

SEPTEMBER 24.

Chairman: Birkenmajer, (and Bortolotti)

1. Prof. Nicolas Saltykov, Belgrade: *Descartes' Work 'la Géometrie' on the 300th anniversary of the 'Discours de la Méthode'.* Research of Loria and G. Milhaud. Criticisms of the conclusions of Milhaud. The evolution of analytic geometry. Eulogy of Descartes by Picard.

2. Prof. Pierre Sergescu, Cluj: *The 'Journal des Savants' and mathematics of the 18th century.* Leibniz, l'Hospital, A. Parent, Rolle, Sorel, Boscovich, Padre Alessandro.

3. Sergescu: *Mathematics in Rumania in the 18th century.* Academy at Jassy 1640, at Bucharest 1678. Hrisant Notara published at Paris in 1716 the oldest book on mathematics written by an inhabitant of the Rumanian principalities. It is a mathematical geography in Latin. Toward the end of the eighteenth century the first mathematical books published in Rumania are circulated: 1777 (anonymous), 1785 the arithmetic of Sincai in Transylvania, 1795 the arithmetic of Bishop Amfilodué de Hotin, 1805 the arithmetic of Obradovics.

4. Louis V. Laurent, Bucharest: *Georges Pachymère, professor of science in the Byzantine University.*

5. Prof. Bogumil Jasinowski, Vilna, Poland: *L'épanouissement de l'atomisme au début du 19e siècle, ses prémisses historiques.* (He was nor present; read by title by Sergescu.)

6. Prof. Arnošt, Dittrich, Prague: *On the Chronology and Astronomy of the Mayas.* The chief question is the correlation of the chronology of the Mayas with ours. There are two proposals, one of Spinden and the other of Thomson. The astronomer H. Ludendorff succeeded in verifying Spinden's correlations. Many dates on the astronomical monuments became significant with its help. The author of the present study maintains Spinden's correlations by means of an analysis of historical accounts, and by astronomical and statistical arguments.

7. Miss Albína Dratvová, Univ. lecturer, Prague: *Newton as Philosopher.* Newton believes firmly in God, whose existence he con-

siders evident and not at all hypothetical. He introduced these notions for logical and religious reasons. It is thus that he came to found a closed system of mechanics before anyone had succeeded in founding one. For Newton as well as for modern physics positivism was a point of methodological and critical departure, but the results which he gave depend on the state of science.

8. Prof. František Kadeřávek, Prague: *Daniel Schwenter-Theophilus Schweighart.* Daniel Schwenter (1585-1636) was a celebrated German scientist, a mathematician and linguist. He wrote a series of scientific books and also, under the pseudonym Schweighart, some Rosicrucian books among which perhaps also the famous *Fama fraternitatis* on which the Rosicrucian Society was founded.

9. Prof. Victor Teissler, Bratislava: *Joseph Petzval.* Petzval was professor of mathematics in Pest, later in Vienna, constructor and calculator of the portrait objective F: 3.6. Two objectives by Petzval were produced in 1844 and 1850.

10. František Křeček, engineer, Prague: *The scientific beginnings of Prof. K. V. Zenger.*

11. Josef Skokan, Prague: *A geographical study of Gregor Mendel.*

SEPTEMBER 27

Chairman: Vollgraff

1. Prof. Gino Loria, Genoa: *Supplementary remarks* on his plenary lecture.

2. Prof. Ettore Bortolotti, Bologna: *Babylonian Mathematics.* Polemic *vs.* Neugebauer. He critized his works, as he has done in his similar numerous publications. Discussion: Léon Rosenfeld, University of Liege, desired to defend Neugebauer. He could not speak in peace, because Bortolotti criticised his explanations after every few sentences. The chairman reminded them that there was insufficient time for a debate. In the general session of September 23 Rosenfeld gave a paper *Remarques sur la question des "précurseurs".* Birkenmajer made a motion that the next Congress (Lausanne, 1940) should make possible a detailed debate on the questions of Babylonian mathematics. Jean Pelseneer, University of Brussels, amended this motion to include not only Babylonian but also Indian and Oriental mathematics.

3. Prof. Johan Adriaan Vollgraff, Leyden: *Christiaan Huyghens' experiments of 1692 on electricity.* Makes use of a "Sphaerula ex electro paulo majore quam pollicari diametro". This paper will be published the end of 1937 in tome 19 of the *Oeuvres Complètes.*

4. Prof. Rudolf Kreutzinger, Brno: Some aspects of the part of geometry in the building of the Middle Ages. Builder's hut and stone-mason fraternity, guild secret, and stone-mason's signs. "Ars sine scientia nihil est."

5. Prof. Bohuslav Hostinský, Brno (Czecho-Slovakia): *The notion of Force in 18th Century Mechanics.* Theory of vortices following the Cartesian, Intuitive definition of Newton, Objections of d' Alembert. Opinions of Boyle, Sauri and Gehler.

6. Prof. Karl Lichtenecker, Prague: *Ch. Doppler and J. Loschmidt.*

There were about 50 foreign participants in the Congress, although 135 had been announced, and 330 others. The distribution of lectures and papers by sections was as follows: General 16, Mathematics 40, Natural sciences 31, Medicine 23, Technology 18, and Agriculture 22. There were 9 plenary lectures and 150 papers announced. America was represented by Dr. Joseph Mayer, consultant at the Library of Congress, Washington, D. C., and Mrs. Mayer. Receptions were cancelled because of the death of President Masaryk. On two afternoons the Congress members made delightful conducted auto drives through the city. A pleasant excursion was made to Karlův Tyn and Křivoklat There were special theater performances and concerts, book, manuscript and instrument exhibitions. A great sesquicentennial celebration of the birth of the Biologist and Physiologist Jan. Ev. Purkyně was held. The delegates found splendid food and lodging.

The Teachers' Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

The Introduction of Invariant Theory Into Elementary Analytic Geometry

By L. E. BUSH
The College of St. Thomas

1. *Introduction.* MacDuffee and Paradiso have called attention to the fact that most American textbooks on Analytic Geometry fail to give complete metric characterizations of second degree curves and surfaces by means of invariants and covariants.* They develop complete systems of invariants and covariants by the methods of the Lie Theory, and by the use of these concomitants they exhibit canonical forms to which every such curve or surface can be reduced, and give a very complete classification into fundamental shapes. The methods used are not, however, very well adapted to class use with elementary students.

Because of the manner in which this interesting topic is presented in the usual textbook, the student often is left ignorant of the power and beauty of the method of invariants. For several years the author has been trying to introduce a more complete treatment of the invariants of second degree curves into his first year analytic geometry courses, and during the past year attempted the same thing in regard to second degree surfaces in a more advanced course in analytic geometry of space. The purpose of the present paper is to give in a general way the method that has been used. Of course, each time the subject is presented, changes are made in the order of presentation and the methods of derivation. In fact, there is room for almost endless variation by the individual teacher.

As has been pointed out by the authors cited above, the chief difficulty encountered in presenting this subject to elementary students is that a complete characterization cannot be made in terms

*C. C. MacDuffee, *American Mathematical Monthly*, v. 33 (1926), pp. 243-252; L. J. Paradiso, *American Mathematical Monthly*, v. 33 (1926), pp. 406-418.

of invariants alone, but certain covariants are also needed. This is probably the reason for its omission from most of the textbooks. The use of covariants can be avoided, however, by the use of certain conditional invariants.* Another difficulty is that of obtaining invariants by some intuitive method. Of course, the teacher can write a function of the coefficients down and proceed to prove its invariance, but the student is left wondering why that particular function was chosen. The author has tried to furnish plausible reasons for the choice of such a function before proving its invariance.

Since the material presented here is intended for first year college students, no knowledge of mathematics is assumed beyond that usually given in a first course in college algebra.† In order to save space, wherever possible proofs and derivations have been omitted and references given to easily obtainable textbooks.

PART I. SECOND DEGREE CURVES

2. *Canonical Forms.* In rectangular coordinates the locus of the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{1}$$

(where the coefficients are all real and at least one of those of the second degree terms is different from zero) is a *second degree curve.* Rotation of the axes through the angle $\frac{1}{2}\text{arc cot}(a-b)/2h$ (or, if $h=0$, no rotation) reduces (1) to

$$a'x^2+b'y^2+2g'x+2f'y+c'=0, \tag{2}$$

where a' and b' are not both zero.‡ If $a'=0$, then $b'\neq 0$ and a further rotation of the axes through 90° reduces the equation to form (2) with $a'\neq 0$. If the origin is now translated to $(-g'/a', y_0)$, (2) becomes

$$a'x^2+b'y^2+2f''y+c''=0, \tag{3}$$

where $a'\neq 0$, $f''=b'y_0+f'$, $a'c''=a'b'y^2_0-g'^2+2a'f'y_0+a'c'$.§ If $b'\neq 0$, the choice of $y_0=-f'/b'$ reduces (3) to

$$a''x^2+b''y^2+c''=0. \tag{4}$$

*The methods used here are, in this respect, similar to those of B. Niewenglowski, *Cours de Géométrie Analytique*, Paris, 1894.

†There is one exception to this. In Part II, in order to get an invariant condition for a surface to be a surface of revolution or a sphere, use is made of the conditions for a cubic equation to have a double or a triple root. This can be omitted without affecting the validity of the rest of the work.

‡R. W. Brink, *Analytic Geometry*, New York 1935, pp. 114-116 and p. 120.

*P. H. Graham, F. W. Johns, and H. R. Cooley, *Analytic Geometry*, New York, 1936, pp. 154-156.

If $b'=f'=0$, (3) has the form (4) for all values of y_0. If $b'=0, f'\neq 0$, the choice of $y_0=(g'^2-a'c')/2a'f'$ reduces (3) to

$$a''x^2+2f''y=0. \tag{5}$$

Thus in every case (1) can be reduced by Euclidean transformation (i. e., by a rigid motion of the axes) to one, and only one, of the two canonical forms

$$Ax^2+By^2+C=0 \qquad (A\neq 0) \tag{6}$$

$$Ax^2+2Fy=0 \qquad (A\neq 0, F\neq 0). \tag{7}$$

3. *Invariants.* It is obvious that there are certain geometrical entities which are dependent only on the curve, i. e., are entirely independent of the particular choice of coordinate system. For example, the length of the major axis of an ellipse is the same, no matter to what axes we refer the curve, so is the angle between the asymptotes of an hyperbola, or the area of an ellipse. In fact, if (1) represents an ellipse, its area is

$$\frac{\pi(abc-af^2-bg^2-ch^2+2fgh)}{(ab-h^2)^{3/2}}.\text{*} \tag{8}$$

Now, if the axes be rotated and translated to a new position, the new equation of the ellipse will in general be entirely different from (1), yet its area is the same and will be given by formula (8), where now we use the coefficients of the new equation instead of those of (1). Thus formula (8) has the property that its value is the same, no matter to what axes the curve be referred. Such an expression is called an ***invariant*** of (1). To be more explicit, if

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0 \tag{9}$$

is the form taken by equation (1) after any series of rotations and translations of the axes, and if $f(a, h, b, g, f, c)$ is a function having the property that $f(a', h', b', g', f', c')=f(a, h, b, g, f, c)$, for all such changes of axes, then $f(a,h,b,g,f,c)$ is called an ***Euclidean invariant*** of (1).

If equation (1) is multiplied through by a constant $k(\neq 0)$, the locus is unchanged. It is easily seen that if the new coefficients were substituted in (8) the only effect would be to multiply both numerator and denominator by k^3, and the value of (8) would remain unchanged. This is not true in general, however, of invariants. MacDuffee calls an invariant the value of which is unchanged when the equation is

*MacDuffee, loc. cit., p. 251.

multiplied through by any constant k a *geometric invariant.** The invariants which we shall consider will be homogeneous functions of the coefficients of (1) hence the effect of multiplying (1) through by k will be to multiply the invariant by a power of k, say by k^m. We shall call such an invariant an *invariant of weight m.*† A geometric invariant is thus an invariant of weight zero. If I and J are invariants of respective weights m and n, it is obvious that I^n/J^m is a geometric invariant.

4. *Derivation of some invariants.* Let us ask for a condition on the coefficients of (1) that it represent a pair of straight lines. The condition is $\Delta=0$, where

$$\Delta=abc-af^2-bg^2-ch^2+2fgh.\ddagger \tag{10}$$

Since the property that (1) be a pair of straight lines is unchanged by a change of axes, we might suspect that Δ is an invariant of (1). That it is can be verified by computation. Let the axes be rotated through an angle θ, after which (1) takes the form (9), where

$$a'=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta,$$

$$h'=(b-a)\sin\theta\cos\theta+h(\cos^2\theta-\sin^2\theta),$$

$$b'=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta,$$

$$g'=g\cos\theta+f\sin\theta,\ f'=-g\sin\theta+f\cos\theta,\ c'=c.$$

Substitution of these values gives, after a somewhat tedious simplification,

$$\Delta'=a'b'c'-a'f'^2-b'g'^2-c'h'^2$$
$$+2f'g'h'=abc-af^2-bg^2-ch^2+2fgh=\Delta$$

Thus Δ is invariant under rotation of the axes through any angle θ. Next, let the origin be translated to (x_0, y_0), after which (1) takes the form (9), where now

$$a'=a,\ h'=h,\ b'=b,\ g'=ax_0+hy_0+g,\ f'=hx_0+by_0+f,$$

$$c'=ax_0^2+2hx_0y_0+by_0^2+2gx_0+2fy_0+c.$$

Again a simple computation shows $\Delta'=\Delta$. Hence Δ is invariant under every translation of axes. Since every rigid motion of the axes

*MacDuffee, loc. cit., p. 247. The name is significant, since it is obvious that only such an invariant can measure a length, angle, area, volume, etc. connected with the curve.

†MacDuffee, loc. cit., p. 247.

‡H. B. Fine and H. D. Thompson, *Coordinate Geometry*, New York 1927, pp. 123-124, give a complete derivation of the answer to the proposed question and show that the condition is both necessary and sufficient.

in the plane can be broken up into a rotation and a translation, Δ is an invariant of (1). Its weight is 3.

We now ask if (1) represents a pair of straight lines, what is the further condition on the coefficients that the lines be parallel (or coincident). If (1) represents parallel lines it can be written

$$(\alpha x+\beta y+\gamma)(\alpha x+\beta y+\gamma')=0,*$$

and $a=\alpha^2$, $h=\alpha\beta$, $b=\beta^2$. If $D=ab-h^2$, it follows that $D=0$.† Since the vanishing of D characterizes a property which is independent of the coordinate system, we are led to ask whether it is an invariant of (1). We can verify that it is in a manner exactly analogous to that used for Δ. Its weight is 2.

Now if (1) represents a pair of straight lines what is the further condition that they be perpendicular? If the lines are $\alpha x+\beta y+\gamma=0$ and $\alpha' x+\beta' y+\gamma'=0$, then $a=\alpha\alpha'$, $b=\beta\beta'$. A necessary and sufficient condition that these lines be perpendicular is $\alpha\alpha'+\beta\beta'=0$, or $a+b=0$. If $L=a+b$, we can once more verify as in the case of Δ and D that L is an invariant of (1) of weight one.

If (1) represents parallel lines we seek the further condition on the coefficients that these lines be coincident. In this case (1) can be written $(\alpha x+\beta y+\gamma)^2=0$, and $a=\alpha^2$, $h=\alpha\beta$, $b=\beta^2$, $g=\alpha\gamma$, $f=\beta\gamma$, $c=\gamma^2$. This gives (among other obvious relations) $ac-g^2=0$ and $bc-f^2=0$, or $M=0$ where $M=ac+bc-f^2-g^2$.‡ Since $M=0$ is characteristic of a geometric property of the curve we are led to suspect that it is an invariant of (1). If we attempt to verify this, we find that it is invariant under rotation of the axes through any angle θ, but it is not invariant under translation. We shall show, however, that it is invariant under translation of the axes in a limited sense, i. e., if $\Delta=D=0$, M is unchanged by a translation. For, suppose $\Delta=D=0$, and that (1) takes the form (9) after the origin has been translated to (x_0, y_0). Then

$$\begin{aligned} M' &= a'c'+b'c'-f'^2-g'^2 \\ &= D(x_0^2+y_0^2)+2(bg-fh)x_0+2(af-gh)y_0+ac+bc-f^2-g^2 \\ &= 2(bg-fh)x_0+2(gh-af)y_0+M. \end{aligned}$$

*α, β, γ, and γ' may be imaginary, even when both lines are real.

†It can be shown by elementary means that $\Delta=D=0$ is a sufficient condition for (1) to represent parallel lines.

‡To see that $\Delta=D=M=0$ is sufficient for (1) to represent coincident lines, we note that $\Delta=D=0$ implies that (1) can be written $(\sqrt{a}x\pm\sqrt{b}y+\gamma)(\sqrt{a}x\pm\sqrt{b}y+\gamma')=0$, whence $f=\frac{1}{2}\sqrt{a}(\gamma+\gamma')$, $g=\pm\frac{1}{2}\sqrt{b}(\gamma+\gamma')$, $c=\gamma\gamma'$. Since a, h, and b are all real and not all zero, $D=0$ implies $ab=h^2\geq 0$, where a and b are not both zero (for then h would vanish also) nor of opposite sign. Hence $a+b\neq 0$. Then $4M=-(a+b)(\gamma-\gamma')^2=0$, implies $\gamma=\gamma'$.

But

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \text{ and } ab - h^2 = 0$$

Multiply the first row of Δ by $-b$ and add to it h times the second row. Expanding by the elements of the first row, we get

$$-b\Delta = (bg - fh)^2 = 0.$$

Similarly, $af - gh = 0$. Hence $M' = M$.

We have now found three invariants of (1), namely L, D, and Δ, and we have found that M, although not an actual invariant, is independent of the choice of axes whenever $\Delta = D = 0$.

5. *Classification of second degree curves.* We have seen that every second degree curve can by an appropriate change of axes be reduced to one, and only one, of the canonical forms (6) and (7). We find that for (6), $L = A + B$, $D = AB$, $\Delta = ABC$, $M = CL$; for (7), $L = A$, $D = 0$, $\Delta = -AF^2$. Let us now subdivide (6) as follows:

(6a) $$Ax^2 + By^2 + C = 0, \ ABC \neq 0.$$

(6b) $$Ax^2 + C = 0, \ A \neq 0.$$

(6c) $$Ax^2 + By^2 = 0, \ AB \neq 0.$$

We note that $\Delta = 0$ for (6b) and (6c), while $\Delta \neq 0$ for (6a) and (7); that $D = 0$ for (6b) and (7), while $D \neq 0$ for (6a) and (6c). Conversely, if $\Delta \neq 0$, $D \neq 0$ for (1), it can be reduced to the canonical form (6a), for it can be reduced to one of the four forms and this is the only one for which $\Delta \neq 0$, $D \neq 0$. Similarly, if $\Delta = D = 0$, (1) can be reduced to (6b); if $\Delta = 0$, $D \neq 0$, it can be reduced to (6c); and finally, if $\Delta \neq 0$, $D = 0$, it can be reduced to (7).

Before proceeding to a more detailed classification, we note that since D and M are of even weight, their signs are not changed by multiplying the equation through by any real constant. This is not true of L and Δ, but it is true of the invariant $L\Delta$, which is of weight 4. Thus the signs of D and $L\Delta$ are characterisitic of the curve, as is also the sign of M when $\Delta = D = 0$.

Now (6a) represents an ellipse or hyperbola according as $AB > 0$ or $AB < 0$, i. e., according as $D > 0$ or $D < 0$. If it is an ellipse, it will be real or imaginary according as C has the opposite or the same sign as A and B, i. e., according as $L\Delta < 0$ or $L\Delta > 0$. In particular, the ellipse is a circle if, and only if, $A = B$, i. e., if, and only if, $L^2 - 4D = 0$.

If (6a) is an hyperbola, it is equilateral if, and only if, $A=-B$, i. e., $L=0$.

Again, (6b) represents a pair of parallel lines. Since $\Delta=D=0$, M is invariant. The lines are real and distinct, coincident, or conjugate imaginary according as $AC<0$, $C=0$, or $AC>0$, i. e., according as $M<0$, $M=0$, or $M>0$ (since $L=A\neq 0$).

Again, (6c) is a pair of intersecting lines. They are real or conjugate imaginary according as $AB<0$ or $AB>0$, i. e., according as $D<0$ or $D>0$. In particular, they are perpendicular if $A=-B$, i. e., if $L=0$.

Finally, (7) is a parabola.

Thus we have a classification of second degree curves by means of conditions which are characteristic of the type of curve and which are easily computed from the equation of the curve referred to any axes whatsoever. Summarizing the classification, we have

I. $\Delta\neq 0$, $D>0$, ellipse.
- (a) If $L\Delta<0$, the ellipse is real.
  If $L\Delta>0$, the ellipse is imaginary.
- (b) If $L^2-4D=0$, the ellipse is a circle, which is real or imaginary according as $L\Delta<0$ or $L\Delta>0$.

II. $\Delta\neq 0$, $D<0$, hyperbola.
- (a) If $L=0$, the hyperbola is equilateral.

III. $\Delta\neq 0$, $D=0$, parabola.

IV. $\Delta=0$, $D<0$, real intersecting lines.
- (a) If $L=0$, the lines are perpendicular.

V. $\Delta=0$, $D=0$, parallel lines.
- (a) If $M<0$, the lines are real and distinct.
- (b) If $M=0$, the lines are coincident.
- (c) If $M>0$, the lines are conjugate imaginary.

VI. $\Delta=0$, $D>0$, intersecting conjugate imaginary lines, sometimes called a point ellipse (if $L^2-4D\neq 0$) or point circle (if $L^2-4D=0$.)

6. *Metric properties.* It is obvious that the coefficients in the canonical forms (6) and (7) are invariants of weight one.* We shall express them in terms of Δ, D, L, and (in the case $\Delta=D=0$) M.

*These canonical forms can be reduced further (by multiplication through by a constant) to forms whose coefficients are geometric invariants. For example, (6a)

For (6a) and (6c), $A+B=L$, $AB=D$. Thus A and B are the roots of $x^2-Lx+D=0$,* while $C=\Delta/D$. For (6b), $\Delta=D=0$, and M is invariant. We have in this case $A=L$, $C=M/L$. For (7), $A=L$, and we may choose for F either root of $Lx^2+\Delta=0$.†

Having determined the coefficients of the canonical forms in terms of invariants which are easily calculated from (1), we may express any metric property of the curve in terms of these invariants. In this sense, Δ, D, L, and (in case $\Delta=D=0$) M completely determine the curve.

For example, the latus rectum of a parabola is $2\sqrt{-\Delta/L^3}$, the distance between parallel lines $2\sqrt{-M/L^2}$, the angles between intersecting lines or between the asymtotes of an hyperbola

$$2 \text{ arc tan } \left(\frac{2D-L^2 \pm L\sqrt{L^2-4D}}{2D} \right)^{1/2}$$

and the area of an ellipse as given by (8) is $\pi\sqrt{\Delta^2/D^3}$.

Consider for instance $x^2-6xy-2y^2+2x-4y+1=0$. Here $L=-1$, $D=-11$, and $\Delta=-1$. The curve is therefore a non-rectangular hyperbola. The angle between its asymtotes is 81°26′. It can be reduced by a change of axes to form (6a), where

$$A=\tfrac{1}{2}(-1+3\sqrt{5}), B=\tfrac{1}{2}(-1-3\sqrt{5}), C=1/11.$$

For $x^2-2xy+y^2+3x-3y+2=0, L=2, \Delta=D=0, M=-\frac{1}{2}$. This is a pair of real parallel lines at a distance of $\sqrt{2}/2$ apart. The canonical form is $2x^2-\frac{1}{4}=0$.

can be multiplied through by $-C/AB$, resulting in the form $\alpha x^2+\beta y^2-\alpha\beta=0$, where $\alpha=-C/B, \beta=-C/A$. The coefficients α and β are roots of the equation $D^3x^2+LD\Delta x+\Delta^2=0$. Since the coefficients of the latter equations are all invariants of weight 6, multiplication of (1) by a constant $k(\neq 0)$ only has the effect of multipling the latter equation through by k^6 and therefore does not affect its roots. Thus α and β are invariants of weight zero. The other canonical forms can be reduced similarly.

*It can be seen from this equation that A and B are invariants of weight one, for L is of weight one and D of weight two. See W. L. Hart, *College Algebra, Alternate Edition*, 1931, p. 235. That we have a choice of either root o. the quadratic as A, and the other as B, is due to the fact that whenever (1) is reducible to (6a) or (6c) it has two axes of symmetry, either of which may be chosen as the new X-axis.

†The choice of values for F is due to the fact that we may have the parabola opening upward or downward in the canonical form.

(To be concluded in December issue in which second degree surfaces will be discussed.)

Mathematical World News

Edited by
L. J. ADAMS

Professor Cornelius Gouwens sends news from the Iowa State College of Ames, Iowa. New appointments include: Gerhard Tintner, Ph.D., Vienna, Assistant Professor; Holly C. Fryer, M. S., University of Oregon, Instructor; Roy H. Cook, M. S., Iowa State College, Instructor. On leave of absence for the academic year 1936-37 is Dr. A. E. Brandt, Senior Statistician U. S. Department of Agriculture, Bureau of Soils. Horace W. Norton, III, M. S., Iowa State College, was appointed Assistant Lecturer at the University of London. He will carry on research work at the Galton Laboratory. Arthur W. Davis, M. S., Iowa State College, was appointed an instructor at the South Dakota State School of Mines. Robert Gage, M. S., Iowa State College, was appointed Statistician at the Mayo Clinic at Rochester, Minnesota.

Professor J. L. Walsh sends the following news items from Harvard University: Professor E. V. Huntington of Harvard University delivered two lectures during July at the third anuual conference of the Cowles Commission for Research in Economics at Colorado Springs; Professor J. H. Van Vleck of Harvard has been appointed Visiting Professor at Princeton University for the first half of the present academic year.

Mr. Dick Wick Hall, University of Virginia, is serving as an instructor in mathematics at the University of California at Los Angeles.

Several Benjamin Peirce Instructorships at Harvard University for the year 1938-39 are open to men who have the degree of Ph.D. or its equivalent. Applications should be sent to the Chairman of the Division of Mathematics.

Additions to the staff of the mathematics department of Rensselaer Polytechnic Institute include: Ralph E. Huston, Ph.D. (Chicago), Dennis Burley Ames, Ph.D. (Yale), and Lynn L. Merrill, Ph.D. (Rensselaer).

The semi-centennial celebration of the founding of the American Mathematical Society will be held September 6-9, 1938 in New York.

In addition to a program of social events there will be nine invited addresses by Professors J. W. Alexander, E. T. Bell, G. D. Birkhoff, G. C. Evans, E. J. McShane, J. F. Ritt, J. L. Synge, T. Y. Thomas and Norbert Weiner.

The annual joint meetings of the American Mathematical Society, the Mathematical Association of America, the American Association for the Advancement of Science, the Association for Symbolic Logic and the Institute of Mathematical Statistics will be held in Indianapolis, Indiana, on December 28-30, 1937.

Professor C. V. Newsom, University of New Mexico, reports that Dr. H. D. Larsen has been promoted to an assistant professorship and that Mr. C. B. Barker, Jr. has been appointed to an instructorship.

Professor B. L. Remick, who has been head of the department of mathematics at Kansas State College for the past thirty-seven years, has retired to a part-time teaching position in the department. Dr. W.T. Stratton, who has been connected with the department for twenty-seven years, took over the active duties as head of the department in September.

Dr. George C. Munro of the University of Michigan became assistant professor of mathematics at Kansas State College at the opening of the present school year, succeeding Dr. Henry Van Engen, who became head of the department at Iowa State Teachers College.

Due to the increased enrollment in mathematics at Kansas State College, Mr. Harold Wierenga was added to the department for full-time teaching during the first semester and for half-time for the second semester.

Professor H. L. Rietz, the State University of Iowa, reports that Dr. Edwin N. Oberg (Ph.D. Minnesota) has been appointed an Instructor of Mathematics at the University of Iowa.

Professor Wm. H. Roever, Washington University (Saint Louis) announces that doctorates involving mathematics were conferred in June, 1937 by that institution as follows:

1. Morris Joseph Gottlieb. Major subject, mathematics. Minor subject, physics. Thesis, *An Investigation of Polynomials Orthogonal on a Finite or Enumerable Set of Points.*

2. Robert Duwayne Miller. Major subject, physics. Minor subject, mathematics. Thesis, *The Effect of Anisotropy in the Atomic*

Vibrations on the Intensity of the Reflections of Mo K X-Rays from Powdered Zinc.

3. Otto Herbert Arnold Schmitt. Major subjects, physics and zoology. Minor subject, mathematics. Thesis, *An Electrical Theory of Nerve Impulse Propagation.*

Professor J. I. Tracey sends these news items from Yale University: Professor W. R. Longley has been appointed acting Dean of Freshmen at Yale University. Dr. Marshall Hall has been appointed an Instructor in Mathematics at Yale University.

Dr. Frank Morley, professor of mathematics at Haverford College from 1887 to 1900, died at his home in Baltimore October 17, aged 77. From 1900 to 1929 he was professor of mathematics at the Johns Hopkins University.

The Bicentennial of the birth of Galvani was celebrated on October 18 at the University of Bologna. Scholars and delegates from many countries were present.

Dr. Richard Philip Baker, since 1916 professor of mathematics at the University of Iowa, died on August 13 at his home in Iowa City, aged 71.

On the occasion of the Bicentennial Jubilee of the University of Göttingen, June 25-30, Dr. David Hilbert, Geheimrat Professor of mathematics emeritus, was presented with the Mittag-Leffler gold medal of the Stockholm Mathematical Institute. Professor Hilbert's health did not permit him to participate in the Jubilee, but the Swedish delegates journeyed to his summer retreat in the Harz mountains for the presentation ceremony. At the same time a similar medal was conferred on the noted Paris mathematician, Prof. Charles Emile Picard, the son-in-law of Hermite. An American great-grandson of Gauss, who attended the Göttingen Jubilee, and as the official (faculty) delegate of his own university spoke for all American universities represented, was greeted most enthusiastically by the audience.

Early in August a special ceremony was held in Paris commemorating the 300th anniversary of the appearance of Descartes' *Discours de la Méthode,* his work on analytic geometry. Prof. Emile Picard delivered the eulogy of Descartes. The United States was represented by Dean Birkhoff of Harvard.

Dr. Carl Louis Ferdinand von Lindemann, professor of mathematics at the University of Munich, celebrated his 85th birthday

on April 12, 1937. In 1882 he proved that π is transcendental. During the years 1870-77 he studied in Göttingen, Erlangen, Munich, London and Paris. Erlangen conferred the Ph.D on him in 1873. Würzburg appointed him to a professorship in 1877; from 1877 to 1883 he was a professor in Freiburg, from 1883 to 1893 at Königsberg. Since 1893 he has been at Munich. He is a member of many scientific societies and has received many honors. On his birthday *Völkischer Beobachter* (Berlin) published a lengthy appreciative article.

Dr. Friedrich Engel, professor of mathematics at the University of Giessen, received from King Haakon VII of Norway the Grand Cross of the Order of St. Olaf, following his 75th birthday on December 26, 1936. This was primarily in recognition of his great services in editing the collected works of Sophus Lie, as well as his historical research. Before going to Giessen he was professor in the Universities of Leipzig and Greifswald.

Volume 3, third edition, of Professor Johannes Tropfke's *Geschichte der Elementar-Mathematik* appeared recently.

Dr. Kurt Mahler has been appointed lecturer on the theory of numbers at the University of Manchester, England.

Dr. Frederick E. Brasch, chief of the Smithsonian Division in the Library of Congress at Washington, will retire January 1, 1938 from the secretaryship of the International History of Science Society, after many years of service. He has been on a research tour of London and the Continent.

Problem Department

Edited by
ROBERT C. YATES

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

SOLUTIONS

Late Solutions. No. 144. *Karleton W. Crain*; No. 147. *A. Moessner.*

No. 138. Proposed by *M. H. Martin*, University of Maryland.

Given a polynomial:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad (a_n > 0)$$

with integral coefficients. The quantity

$$h = n + a_n + |a_{n-1}| + \cdots + |a_1| + |a_0|$$

is known as the *height* of the polynomial.

(1) Find a formula for the number of polynomials possessing a given height h.

(2) Find a formula for the number of polynomials possessing a given height h and a given degree n.

(3) Show that for polynomials having a fixed height h the number of polynomials of degree n equals the number of degree $h-n-1$.

Solution by the Proposer.

The number $N(h,n)$ of polynomials of degree n and height h is given by the formula

$$N(h,n) = 1 + 2\binom{n}{1}\binom{h-n-1}{1} + 2^2\binom{n}{2}\binom{h-n-2}{2} + \cdots.$$

To prove this formula, write the relation defining the height h in the form of a Diophantine equation

$$a_n+|a_{n-1}|+\cdots+|a_0|=h-n, \qquad a_n>0,$$

where $h-n>0$. Consider the product

$$(\xi'+\xi^2+\cdots+\xi^{a_n}+\cdots)(\xi^0+2\xi'+2\xi^2+\cdots+2\xi^{|a_{n-1}|}+\cdots)\cdots$$
$$(\xi^0+2\xi'+2\xi^2+\cdots+2\xi^{|a_0|}+\cdots).$$

The coefficient of

$$\xi^{h-n}=\xi^{a_n}\xi^{|a_{n-1}|}\cdots\xi^{|a_0|}$$

in this product equals the number $N(h,n)$ of distinct solutions of the above Diophantine equation. Writing this product in the form

$$\frac{\xi}{1-\xi}\left(\frac{1+\xi}{1-\xi}\right)^n=\frac{\xi}{1-\xi}\;\frac{(1-\xi+2\xi)^n}{(1-\xi)^n}$$
$$=\xi\left[\frac{1}{1-\xi}+\binom{n}{1}\frac{2\xi}{(1-\xi)^2}+\binom{n}{2}\frac{2^2\xi^2}{(1-\xi)^3}+\cdots\right],$$

the above expression for $N(h,n)$ is readily obtained.

The remaining parts of the problem are now readily secured on the basis of this result. See a paper about to appear in the L'Enseignment Mathématique entitled *Sur le nombre de polynômes à hauteur donnée* by M. H. Martin and A. Seidenberg, where an alternative derivation is also given.

No. 153. Proposed by *R. E. Gaines*, University of Richmond, Virginia.

A variable line is tangent to $\rho=a(1+\cos\Theta)$ at P and intersects the curve in two other points, Q and R. Prove that the locus of the midpoint of the chord QR is a limacon whose equation (with suitable change of origin) is $2\rho=a(1+4\cos\Theta)$.

Solution by the Proposer.

Let the equation of the tangent to

(1) $$\rho=a(1+\cos\Theta)$$

at P be:

(2)* $$\cos(\Theta-3\Theta_1/2)=2a\cos^3\Theta_1/2$$

*See, for instance, Granville, Smith, and Longley's *Calculus* (1929), p. 125.—Ed.

Eliminating Θ between (1) and (2), remembering that ρ_2 is a double root of the resulting quartic in ρ, we have

(3) $$2\rho^2-4(a-\rho_1)\rho+a\rho_1=0$$

whose roots are ρ_2 and ρ_3. Thus, directly,

(4) $$\rho_2+\rho_3=2(a-\rho_1),\quad 2\rho_2\rho_3=a\rho_1,\quad \rho_2^2+\rho_3^2=4a^2-9a\rho_1+4\rho_1^2$$

For M, the midpoint of P_2P_3, we have

$$2\rho\cos\Theta=\rho_2\cos\Theta_2+\rho_3\cos\Theta_3=(\rho_2^2+\rho_3^2-a\rho_2-a\rho_3)/a, \qquad \text{by (1)}$$

and, by using (4):

(5) $$2a\rho\cos\Theta=4\rho_1^2-7a\rho_1+2a^2$$

The corresponding equation for $2a\rho\sin\Theta$ may be obtained but the reduction involves long transformations. We obtain a second equation more easily as follows. From equations (1) and (2) eliminate ρ to find:

(6) $$2a\cos^2\Theta+4\rho_1\cos\Theta+5\rho_1-2a=0$$

Thus: $2a\cos\Theta_2\cos\Theta_3=5\rho_1-2a$ and $2a\sin\Theta_2\sin\Theta_3=3\rho_1$

whence $$a\cos(\Theta_2-\Theta_3)=4\rho_1-a$$

The length of the chord P_2P_3 is given by the equation

(7) $$(P_2P_3)^2=\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos(\Theta_2-\Theta_3)=4a^2-8a\rho_1.$$

But we have:

(8)* $$(P_2P_3)^2+(2\rho)^2=2\rho_2^2+2\rho_3^2$$

and therefore

(9) $$2\rho^2=4\rho_1^2-5a\rho_1+2a^2.$$

Eliminating ρ_1 from (5) and (9), we have

(10) $$4(\rho^2-a\rho\cos\Theta)^2-5a^2(\rho^2-a\rho\cos\Theta)-2a^2\rho^2+2a^4=0.$$

Setting $\Theta=0$, equation (10) becomes

$$(\rho-a)(\rho-2a)(2\rho+a)^2=0$$

which gives the double point of the curve at $\rho=-a/2$.

With this point as new origin equation (10) becomes

$$\rho^2(4\rho^2-16a\rho\cos\Theta+16a^2\cos^2\Theta-a^2)=0$$

*Since the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the four sides.

or $$2\rho = a(1+4\cos\Theta).$$

Editor's Note. The character of this problem seems to indicate a solution by the methods of Inversive Geometry. The cardioid $\rho=1+\cos\Theta$ (setting $a=1$ for convenience) has for its *map* equation:*

(1) $$x=2t+t^2$$

where x is a complex variable and $t=e^{i\Phi}$ a parameter. Its line equation is

(2) $$x-3t-3t^2+t^3\bar{x}=0,$$

$\bar{x}$ being the conjugate of x, the running coordinate on the tangent line. For a fixed t, say $t=b$, this is a particular tangent, cutting the curve again in x_2 and x_3, two extra points whose *parameters* t_2, t_3, are obtained by eliminating x between (1) and (2). We find:

(3) $$(t-b)^2\cdot[t^2+2(1+b)t+b]=0$$

the second factor yielding the roots in question, t_2, t_3. The midpoint of x_2, x_3 is, from (1) and (3):

$$x=(x_2+x_3)/2=t_2+t_3+(t_2^2+t_3^2)/2,$$

or

(4) $$x=b+2b^2, \qquad \text{(where } b=e^{i\Phi}\text{)}$$

which is, for variable b the map equation of a limacon. Its rectangular equation in parametric form may be had by placing $u+iv$ for x and $\cos\phi+i\cdot\sin\phi$ for b and equating reals and imaginaries.

It is obvious from (4) that the vector x may be replaced by $k\cdot x$, with k real, to give a limacon of different size. Thus *every* point of division of the segment of the tangent traces out a limacon.

No. 154. Proposed by *W. V. Parker*, Louisiana State University.

Three isosceles triangles with base angles α, β, γ are constructed on the sides a, b, c, of a triangle as bases. If $\alpha+\beta+\gamma=\pi/2$ and D, E, and F are the vertices of the isosceles triangle, prove that the angles of the triangle DEF are $\pi/2-\alpha$, $\pi/2-\beta$, $\pi/2-\gamma$.

Solution by the Proposer.

Let the triangle be so lettered that angles at B and C are both acute. Rotate the triangle DCE about E until C coincides with A and D falls at D_1. Note that this may be done by rotating in one

*See *Inversive Geometry*, Morley and Morley, Ginn (1933), pp. 239-240.

direction through an angle equal to $\pi - 2\beta$. Rotate the triangle DFB about F until B coincides with A and D falls at D_2. Now, angles

$$FAE = A + \beta + \gamma, \quad EAD_1 = C + \alpha + \beta, \quad D_2AF = B + \alpha + \gamma.$$

Therefore, the sum of these three angles is 2π, so that angle $D_1AD_2 = 0$, and since $BD = CD$, D_2 coincides with D_1.

Triangles DEF and D_1EF are congruent since the three sides of one are equal respectively to the three sides of the other. But

$$D_1ED = \pi - 2\beta, \quad \text{and} \quad DFD_1 = \pi - 2\gamma,$$

the angles through which the triangles were rotated. Therefore,

$$FED = (D_1ED)/2 = \pi/2 - \beta, \quad DFE = (DFD_1)/2 = \pi/2 - \gamma,$$

and hence $$EDF = \beta + \gamma = \pi/2 - \alpha.$$

Also solved by *J. Rosenbaum.*

No. 156. Proposed by *Walter B. Clarke*, San Jose, California.

Under what conditions will the line joining the incenter and the verbicenter of a scalene triangle be perpendicular to a side?

Solution by *Karleton W. Crain*, Purdue University.

In trilinear coordinates the equation of the line joining the incenter and the verbicenter is:

$$\alpha \cdot (\sin C \cdot \sin A - \sin A \cdot \sin B) + \beta \cdot (\sin A \cdot \sin B - \sin B \cdot \sin C) + \gamma \cdot (\sin B \cdot \sin C - \sin C \cdot \sin A) = 0.^*$$

The equation of the side BC is $\alpha = 0$.

The condition that these two lines be perpendicular is,

$$(\sin C \cdot \sin A - \sin A \cdot \sin B) - (\sin B \cdot \sin B - \sin C \cdot \sin A)\cos B - (\sin A \cdot \sin B - \sin B \cdot \sin C)\cos C = 0.^\dagger$$

Using the Law of Sines and the Cosine Law, this condition reduces to

$$(c-b) \cdot [3a^2 - (b^2 + 2bc + c^2) + 2a(b+c)] = 0,$$

or $$(c-b)(a+b+c)(3a-b-c) = 0,$$

*Cf. solution of problem No. 90, N. M. M., November, 1935.
†See, for instance, Smith *Conic Sections*, page 347.

only one of whose factors may vanish. That is

$$3a-b-c=0.$$

Thus, the line in question will be perpendicular to a side if the sum of two sides of the triangle is three times the third side.

Also solved by *C. E. Springer* and the Proposer.

PROPOSALS

No. 170. Proposed by *Walter B. Clarke*, San Jose, California.

Using the notation:

I for incenter,
O for circumcenter,
H for orthocenter,
V for verbicenter,
R for circumradius,
r for inradius,

of a plane triangle, show that:

$$(IV)^2+(IH)^2=4\cdot OV\cdot(R-r).$$

No. 171. Proposed by *Walter B. Clarke*, San Jose, California.

Considering only plane, obtuse, scalene triangles with sides and areas integral:

(1) What are the sides of the one of least area?

(2) What are the sides of the smallest two triangles having equal areas but all six sides different?

(3) Of those having areas equal numerically to perimeters, what is the one of least area?

(4) What is a value of x such that two triangles of equal area have their semi-perimeters equal to x^2+x and x^2-x, respectively?

No. 172. Proposed by *Walter B. Clarke*, San Jose, California.

Show that if the X-line (the line segment joining incenter to verbicenter) is parallel to one side of a triangle then the sides form an

arithmetic progression, the constant difference being the length of the X-line.

No. 173. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Two variable spheres (X), (Y) cut a fixed plane along the same circle, real or imaginary. (A) and (B) are two concentric spheres, one orthogonal to (X), the other to (Y). The center X of (X) describes a fixed curve (F). Find the locus of the center Y of the sphere (Y).

No. 174. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

If the circumdiameters of the tetrahedron $ABCD$ passing through the vertices A, B, C, D meet the sphere again in the points P, Q, R, S, respectively, and the corresponding faces of $ABCD$ in the points K, L, M, N, we have

$$\frac{KP}{AK}+\frac{LQ}{BL}+\frac{MR}{CM}+\frac{NS}{AN}=2.$$

No. 175. Proposed by *G. W. Wishard*, Norwood, Ohio.

Show that any integer in an odd scale is odd, if and only if it has an odd number of odd digits. For example, 32400 in the quinary system is odd, because it has one odd digit. Again, 1357 in the nonary system is even, because it has an even number of odd digits, and is therefore divisible by 2.

No. 176. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Find all positive integral values of n which satisfy the equation:

$$(2+\sqrt{2})^n+(2-\sqrt{2})^n+2^n+2^{2n-1}=\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{n!}\cdot 2^{n+2}.$$

No. 177. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

It is known that if two circles satisfy the condition:

$$1/(R+d)+1/(R-d)=1/r,$$

where R, r, d are the respective radii and the distance between the centers, then any point on the outer circumference is a vertex of a triangle of which the two circles are respectively the circumcircle and

the incircle. Also, that if two circles are respectively the circumcircle and the incircle of a triangle then they satisfy the above equation.

Prove that the corresponding equation for a quadrilateral is:

$$1/(R+d)^2+1/(R-d)^2=1/r^2.$$

No. 178. Proposed by *V. Thebault*, Le Mans, France.

The point M is arbitrarily chosen on the circumcircle of the hexagon $A_1, A_2, A_3, A_4, A_5, A_6$, whose opposite angles are equal or supplementary. Let $B_1, B_2, B_3, B_4, B_5, B_6$ be the orthogonal projections of M on the sides $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1$. Show that the sides of the hexagon $B_1, B_2, B_3, B_4, B_5, B_6$ are parallel three by three to two determined directions.

No. 179. Proposed by *V. Thebault*, Le Mans, France.

Find a perfect square of five digits such that the eight digits with which the number and its square root are written, in the system of base 8, are all different.

No. 180. Proposed by *V. Thebault*, Le Mans, France.

Determine the parallels $\Delta_a, \Delta_b, \Delta_c$ to the sides BC, CA, AB of a triangle ABC which meet AB and AC, BA and CB, CA and CB respectively in M_a and N_a, M_b and N_b, M_c and N_c such that the circles circumscribed to the triangles AM_aN_a, BM_bN_b, CM_cN_c are orthogonal two by two. Show that the radical center of these circles is on the Brocard diameter of the triangle ABC.

No. 181. Proposed by *A. Gloden*, Luxembourg.

Find a perfect square of nine digits of the sort:

$$abcdefghi$$

such that

$$\begin{cases} abc=u^2+4 \\ def=v^2 \\ ghi=w^2+13. \end{cases}$$

No. 182. Proposed by *A. Moessner*, Nurnberg-N, Germany.

What is the solution of the identities:

$$\begin{array}{ll} A_1\cdot A_2 & =A_1+A_2 \\ A_1\cdot A_2\cdot A_3 & =A_1+A_2+A_3 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ A_1\cdot A_2\cdot A_3\cdots A_x & =A_1+A_2+A_3+\cdots+A_x\ ? \end{array}$$

Reviews and Abstracts

Edited by
P. K. SMITH

The Handmaiden of the Sciences. By Eric Temple Bell. The Williams & Wilkins Company, Baltimore, 1937. viii+216 pages. $2.00.

This is a little book on a very big subject and treats briefly a large number of mathematical topics. It presupposes very little technical knowledge and the reader should not expect to secure much such knowledge from it in view of the small amount of space devoted to each subject. Its object seems to be to inspire the reader, but it fails to provide much assistance to the reader who might become interested in securing additional information along various lines. It contains a brief table of contents but no index. Fortunately its author is a man of high mathematical standing and gives his views with great clarity even if they may sometimes be biased.

Among the subjects which are briefly explained are the following: continuity, discreteness, rates, higher derivatives, partial derivatives, graphs, what Descartes invented, curvature, invariance, integrals, the calculus of variations, groups, matrices, probability, statistics, and mechanics. All this and much more on less than 216 pages. When I read what is said about groups, for instance, I wondered whether the reader would find enough there to give him a clear enough idea to be really useful and whether he would not have profited by references to a few places where he could extend his knowledge. The author refers however on page 183 to an important application of substitution groups to mathematical physics since 1925, and such references to recent advances are important features of the book.

The book contains a considerable number of historical references. Inaccuracies along this line should perhaps not be regarded as very serious since the historical matter seems to have been introduced for the purpose of maintaining an interest in the other subjects treated. Instead of saying on page 20 that Menaechmus "invented the conic sections" it would have been better to say that this view had been expressed by some writers. Various statements relating to Descartes' work appear to the reviewer as very misleading. For instance, on page 91 it is said that Descartes "took the initial step of translating geometry into algebra" while on page 100 of volume 6 (1924) of J. Tropfke's ***Geschichte der Elementar-Mathematik*** it is said that the noted

French mathematician F. Vieta created algebraic geometry shortly before Descartes was born. Both geometric algebra and algebraic geometry are older than Descartes.

The remarks relating to the influence of Descartes in regard to the coördinate systems are also apt to give an incorrect impression since such systems are in part much earlier than the work of Descartes. It would be difficult to find any important improvement on a coördinate system which is known to have been due to Descartes. The influence of Descartes on the development of analytic geometry is universally acknowledged as having been very great but it is somewhat difficult, if not impossible, to give a brief and accurate account of it which can be comprehended by those who have not made a special study of the subject. Among the obvious misprints we may refer to the dates of birth of L. Euler and J. J. Sylvester as 1702 and 1815 instead of 1707 and 1814 on pages 40 and 195, respectively.

University of Illinois. G. A. MILLER.

A First Year of College Mathematics. By R. W. Brink. D. Appleton-Century Company, New York, 1937. xxvi+666 pages.

This text presents a new organization of college algebra, analytic geometry, and trigonometry combined. The book is not divided into these separate units, though it would be easy to pick sequences of chapters in such a way as to give a thorough course in each. Coördinates and graphs, and function concept unify the course as presented. Since some analytic geometry is ordinarily included in each of the other subjects, this text eliminates useless duplications.

The simpler portions of analytics are presented first, then some algebra, some trigonometry, then more analytics and the more difficult portions of algebra. The following sequence of chapters (21 through 25) will give some indication of the arrangement: the "Inverse Trigonometric Functions", "Polar Coördinates", "Complex Numbers", "Theory of Equations", "The Straight Line."

For an eight or ten hour freshmman course the book would be especially valuable. The typography of the book is excellent.

Almost every paragraph and problem from Professor Brink's 1933 *College Algebra* and his 1928 *Plane Trigonometery* are incorporated in the new text with only minor revisions aside from order of presentation; several of the earlier chapters of *College Algebra* are placed in the appendix. On the other hand, there is considerable revision of the *Analytic Geometry* as compared with the author's 1924 text. Practi-

cally every problem is new, and the presentation of theory is better adapted to the unified course.

Belhaven College. DOROTHY McCOY.

The Place of Mathematics in Modern Education. The eleventh yearbook of the National Council of Teachers of Mathematics. Bureau of Publications, Teachers College, Columbia University, New York 1937. vi+257 pages.

The contents are arranged in eight main divisions covering 258 pages. Writers of the different divisions are: W. D. Reeve, William Betz, E. T. Bell, David Eugene Smith, Sir Cyril Ashford, W. Lietzmann, Georg Wolff, and Griffith C. Evans. Two of these may be considered as mathematical specialists and authorities in applied fields while the others are outstanding educators and supervisors in America, England, and Germany. As stated in the Preface the purpose is to clarify some of the most important issues with respect to the importance of mathematics in the schools. One should not be surprised to find that it actually does far more than that.

We are all aware of the agitation which gained considerable momentum during recent years of depression in favor of a secondary school curriculum based on so-called social and economic utilities. The underlying philosophy is based on experimental contacts with such social and industrial activity as may be accessible to pupils, building instruction around current social practice by creating so-called real life situations in the school as responses to felt needs. Present curricula which introduce instruction in broad fields of knowledge such as language, history, mathematics, and the sciences would be supplanted by a variety program which would bring in items from these sources only as they supplied an immediate need for the project at hand. The issues referred to above revolve about the twofold question of whether such could be done and whether the outcome would be best. It appears to the reviewer that such issues need clarifying.

It would be absurd to attempt here to give an adequate statement of the content of the volume. We have not seen a statement of both sides of the curriculum problem that would even remotely compare with the Eleventh Yearbook. The purpose of this review is to give enough information to induce one to read the volume before he proceeds with any scheme of curriculum revision. To do this we present each main division in order.

I. *Attacks on Mathematics and How to Meet Them*—W. D. Reeve, pp. 1-21.

The writer begins with 15 critical statements which are typical of the many we have seen reflecting both honest and biased opinion. For example: "It (meaning mathematics) never gets back into life", "I cannot see that algebra contributes one iota to a pupil's health or one grain of inspiration to his spirit. I can see no use for it in the home as an aid to a parent, a citizen, a producer, or a consumer", "I cannot think of a single tough spot in my existence in which Euclid reached down to lend a helping hand". On the other side we quote the following as typical: "The solution of the problem lies in better trained teachers and not in the elimination of fundamental fields of knowledge", "I can organize my ideas more logically, better avoid overstatements, and express thought more accurately, because I did six years of mathematics during my term in college", "Mathematics is no longer content to be a tool; it threatens to absorb almost everything else", the statement that Thales, Pythagoras, Plato, Tartaglia, D'Alembert, Eisenstein, Boole, Fermat, and Sir Isaac Newton would have been missed by not requiring mathematics.

Among the conclusions are the following: 1. A mathematics teacher should have completed a substantial course in the calculus. 2. We must educate the educators regarding the importance of mathematical education and show them a new model in teaching. 3. College mathematics departments should take a more prominent part in mathematics teacher training.

II. *The Reorganization of Secondary Education*—William Betz, pp. 22-135.

One could hardly imagine a more comprehensive survey of the present situation than that contained in these 113 pages. The problem is introduced by ten questions which are treated in six main divisions. Education is viewed as a unified process leading from the kindergarten through the university. Referring to present world wide unrest the situation is viewed as philosopher, psychologist, scientist, and educator. Quotations from various sources abound in such quantity as to constitute a source for wide reference. Among the basic philosphies behind the many conflicting views are, Pragmatic Instrumentalism, Change and Social Reconstruction, Permanent Values, Progressive Education, Curriculum Solutions, Planless Schools, The Core-curriculum, Integration, and Basic Frames of Reference. We find on page 77 the following remarkable statement: "Hence the permanent

elements of the curriculum must be those natural, mental, and spiritual adaptations which race experience has shown to be essential for human progress and for an integrated social life. We must have at least six such elements in the elementary curriculum as follows: practical arts, language, science, mathematics, social studies, and fine arts." Following the statement we find mathematics selected for a five-page statement of importance. Samples of the content are the following: No other country feels called upon to offer a defense of mathematics. It will remain a core-subject long after the vagaries of our day have disappeared. However many of our boys and girls are being deprived of an essential element in education by a misdirected propaganda against mathematics. Seventeen states have already omitted mathematics from the list of prescribed subjects. Reasons given by reformers are that it is not useful nor practical and yet the universities cannot admit students to the practical fields of agriculture, commerce, economics, pharmacy, pre-medicine, geography, or geology without the mathematics requirement. A strange paradox! There must be some other reason for this movement in which America stands alone. We should at least face the question of what civilization would do with a generation of people who were denied the study of mathematics in the public schools.

Following a discussion of mass education, vocational training, standards, and teacher training we find six summary conclusions which are too long to quote. The writer rings the bell and one must read it to appreciate it.

III. *The Meaning of Mathematics*—E. T. Bell, pp. 136-158.

We find here, in the usual refreshing style of the writer, a plain statement of present conditions. It includes a plea for distinction between mathematics and alleged uses. We are urged to teach mathematics as mathematics and uses as uses. Lest we remain in doubt, illustrations of mathematical systems are given. The writer speaks of the scientific and industrial revolution which we now enjoy, wherein the blind appliers of formula and yard stick are out of a job. Under actual and potential uses we find science, economics, finance, games of reason, and various types of argumentation. He designates as objective the molding of the mind to a scientific age.

In conclusion we have four suggestions. 1. Reject all antiquated texts. 2. Review critically all new texts. 3. Have mathematics teachers write texts deductively sound. 4. Aim all elementary mathematics at the calculus as a goal. Note the contrast between this view

and that of those who are eliminating from the curriculum the small amount of mathematics which still remains.

IV. *The Contribution of Mathematics to Civilization*—D. E. Smith, pp. 159-181.

The nature of mathematics interwoven into all phases of our civilization is condensed within a few pages. The knowledge and wisdom of many years are reflected in most attractive form. Like a benediction the conclusion gives the seven lights which mathematics sheds upon civilization. They are: 1. The light of universal utility. 2. Beauty. 3. Imagination which leads to discovery. 4. The poetic mystery which leads to a great experience. 5. The mystery of the unattained. 6. The infinite. 7. The light of religious philosophy.

V. *The Contribution of Mathematics to Education*—Sir Cyril Ashford, pp. 182-193.

This article is a critique of teaching methods in England. The writer stresses teaching the mathematics and using applications as means of objectifying and illustrating the meaning and importance of the mathematics. There is no intimation of a desire to abandon mathematics in England, but rather to stress better teaching.

VI. *Mathematics in General Education*—W. Lietzmann, pp. 194-206.

At the lower level the movement in Germany is to capitalize on mathematics for the masses by emphasis on varied uses. The writer specifies at great length certain important values for the merchant, artisan, artist, scientist, citizen, necessary for intelligent interpretation of society in Germany and its relation to other countries. Specifications include space perception, form, continuous change, and proportions; quantitative thinking with reference to value, magnitude, and relation; the function, the graph, and the limit property of motion; language of concepts, sequences, responsibilities, combinations, and scientific systems. The tendency in Germany is to exploit the values of mathematics for the masses with a view to competing with societies of other nations.

VII. *Mathematics as Related to Other Great Fields of Knowledge*—Georg Wolff, 207-244.

This section stresses the other German view that mathematics is taught for its relation to other broad fields. Thirty-seven pages are devoted to mathematics in natural science, physics, chemistry, botany, zoology, music, painting, sculpture, architecture, and poetry. One

must recognize the contrast between the European program and the present tendency in our schools.

VIII. *Form and Appreciation*—Griffith C. Evans, pp. 245-258.

This address delivered before a convention of Phi Beta Kappa is a fitting climax to a volume devoted to truth regarding the "Queen of the Sciences". It is descriptive, illustrative, and inspiring.

Ordinarily a review should not be concluded without some criticism. Adopting this frame of mind let us complain that the Eleventh Yearbook is of such character that even a long review fails to review it. Let us hope that the effort may create an urge to read. Parent-teacher Associations of America should use it for a series of discussions regarding the interests of their children. Instead they will probably discuss daughter's sinus and son's damaged thumb. Teachers will let them get away with it while they are busy adding to the Museum of Courses now on exhibit at Columbia University. At the last reading the Museum was alleged to have more than thirty five thousand proposed courses.

Mississippi State College. C. D. SMITH.

CORRIGENDA

In Vol. XI, No. 7, April, 1937, the following corrections should be made:

(a) The last equation on page 305 should read

$$2x(x_3-x_2)(y_3-y_2)+2y(y_3-y_2)^2-y_2(y_3-y_2)^2$$
$$+(x_3-x_2)(x_2y_2+x_3y_2-2x_2y_3)=0.$$

(b) The denominator of the $y-$coordinate of N_4, page 307, should read $4(y_3-y_2)$.

(c) The equations in the last two lines preceeding Theorem II, page 307, should be omitted.